Indian Statistical Institute
M. Math. II Year
Mid-Semestral Examination 2008-2009
Fourier Analysis

Date: 23-09-2008 Total Marks: ≥ 50

Maximum marks you can get is 50.

1. (a) Let $f \in C^p(S' \times S' \times \cdot \times S') = C^p(S^n)$ for some $p \geq \frac{n+1}{2}$. Show that [4]

$$\sum_{k \in Z^n} |\hat{f}(k)| < \infty.$$

(b) Show that $\sum\limits_{|k| \leq L} \hat{f}(\underset{\sim}{k}) \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{\theta}}}{(2\hat{u})^{n/2}}$ converges uniformly to $f(\underset{\sim}{\theta})$ as $L \to \infty$.

[2]

2. (a) Show that closure (in the sup norm) of linear span [3]

$$\{e^{i\underset{\sim}{k}\cdot\underset{\sim}{\theta}} : \underset{\sim}{k} \in Z^2, \underset{\sim}{\theta} = (\theta_1, \theta_2)\} = C(S' \times S').$$

(b) Show that $\{f \in C([0, 2\pi] \times [0, 2\pi] : f(0, t) = f(2\pi, t), f(s, 0) = f(s, 2\pi)$ for all $s, t\}$ is dense in $L^2([0, 2\pi] \times [0, 2\pi])$. You can assume $C([0, 2\pi] \times [0, 2\pi])$ is dense in $L^2([0, 2\pi] \times [0, 2\pi])$. [3]

(c) State and prove a theorem connecting $\|f\|_{L^2([0,2\pi]^2)}$ and $\sum\limits_{\underset{\sim}{k} \in Z^2} |\hat{f}(\underset{\sim}{k})|^2$ when $\hat{f}(\underset{\sim}{k})$ is the Fourier coefficient of f. [3]

3. Let $f_0 = \chi_{[0,1)}$ and $f_{j,n} = \chi_{[j2^{-n},(j+1)2^{-n})} - \chi_{[(j+1)2^{-n},(j+2)2^{-n})}$ for $0 \leq j \leq 2^n - 2$, $n = 1, 2, 3, \ldots$. If $g \in L^2[0, 1]$ is orthogonal to f_0 and $f_{j,n}$ for all j, n show that $g \equiv 0$. [3]

4. a) Let $f(x) = e^{-x^2/2}$ for x real. Show that $\hat{f} = f$. [2]

   b) Let $\alpha > 0$. Let $g(x) = e^{-\alpha x^2}$ for x real calculate $\hat{g}$. [1]

c) Let g be as in (b). Show that g attains the minimum in Heisenberg Uncertainity principle. [4]

d) Let $\epsilon_k = \text{linspan}\{e^{-\frac{x^2}{2}}x^j : 0 \leq j \leq k\}$. Show that $\hat{\epsilon}_k = \epsilon_k$ for $k = 0, 1, 2, \ldots$. [3]

5. (a) Let $f(x) = x^{-1/2}$ for $0 < x \leq 1$ and 0 otherwise. Find $(M_f)(0)$. [1]

   b) If $g : R \to \mathbb{C}$ is a measurable function such that $g(x) = 0$ for $x \leq 0$ and $\lim\limits_{x\to\infty} |g(x)| = \infty$, show that $(M_g)(u) = \infty$ for all u. [4]

6. Show that [3]

$$HL'(R) \not\subset L'(R).$$

7. Prove Marcinkiewics theorem viz. let $(X, \mathcal{S}, \mu), (Y, \Gamma, \gamma)$ be σ-finite measure spaces. Let $\mathcal{F}(X, \mathbb{C}) = \{f : [X, \mathcal{S}] \to \mathbb{C},\ f \text{ is measurable}\}$. Let T be a sublinear operator: $\mathcal{F}(X, \mathbb{C}) \to \mathcal{F}(Y, \mathbb{C})$ satisfying
   (a) If $g \in L^\infty(X, \mu)$, then $Tg \in L^\infty(Y, \gamma)$ and there exists a constant C_∞ such that

$$\|Tg\|_\infty \leq C_\infty \|g\|_\infty.$$

   (b) Let $1 \leq p_o < \infty$. If $g \in L^{p_0}(X, \mu)$ then Tg satisfies

$$t^{p_0}\gamma\{y : |(Tg)(y)| > t\} \leq C_{p_0}\ \|g\|_{p_0}^{p_0}$$

   for some constant c_{p_0} independent of g.

   Then for $p_0 < p < \infty$, there exists a constant C_p such that $\|Tg\|_p \leq C_p\ \|g\|_p$ for all $g \in L^p(X, \mu)$. [10]

8. Show that $L^p_w(R^n) \not\subset \bigcup\limits_{q\geq 1} L^q(R^n)$. [4]

9. Let $f = \chi_{[0,1]}$. Calculate $\hat{f}$. Show that (a) $\hat{f} \in \bigcap\limits_{p>1} L^p(R)$ [2],

   (b) $\hat{f} \notin L'(R)$. [2]

10. Let $f \in L'_{\text{loc}}(R^n)$. If $M_f \in L'(R^n)$, then $f \equiv 0$. [3]